Information Fusion using Belief Functions

New combination rules

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Philippe Smets (1938-2005)

Overview

1. Theory of belief functions
   - Motivations
   - Basic concepts
   - Canonical conjunctive decomposition
2. The cautious and bold rules
   - Informational orderings and the LCP
   - The cautious conjunctive rule
   - The bold disjunctive rule
3. Families of combination rules
   - T-norm-based rules
   - Uninorm-based rules
   - Applications

Belief functions
An uncertainty representation framework

- One of the main frameworks for reasoning with partial (imprecise, uncertain) knowledge, introduced by Dempster (1967) and Shafer (1976)
- Belief functions generalize:
  - probability measures;
  - crisp sets;
  - possibility measures (and fuzzy sets).
- Different semantics for belief functions:
  - Lower-upper probabilities (Dempster's model, Hint model);
  - Random sets;
  - Degrees of belief (Transferable Belief Model - TBM).
- The latter model will be adopted in this talk.

The Transferable Belief Model

An interpretation of belief function theory

- A subjectivist, non probabilistic interpretation of Belief Function Theory introduced by Smets (1978-2005).
- Main features:
  1. Semantics of belief functions as representing weighted opinions of rational agents, irrespective of any underlying probability model;
  2. Distinction between the credal and pignistic levels, and use of the pignistic transformation for mapping belief functions to probability measures for decision-making.
  3. Use of unnormalized mass functions and interpretation of $m(\emptyset)$ under the open-world assumption;

Information fusion in the TBM framework

- In recent years, there has been may successful applications of the TBM to information fusion problems (sensor fusion, classification, expert opinion pooling, etc.);
- However, there is some lack of flexibility for combining information as compared to other theories such as Possibility Theory:
- Only two main operators:
  - TBM conjunctive rule $\bigcirc\!\!\!\!\!\cap$ (unnormalized Dempster's rule);
  - TBM disjunctive rule $\bigcirc\!\!\!\!\!\cup$;
- Main limitations:
  - Undesirable behavior of Dempster's rule in case of high conflict between sources;
  - These operators assume the sources to be distinct.

Problem of conflicting evidence

- Many research works devoted to this problem.
- Several alternatives to Dempster's rule based on various schemes for distributing the mass $m(\emptyset)$ to various propositions (Dubois-Prade rule, Yager's rule, etc).
- Some of these rules may be more robust than Dempster's rule in case of highly conflicting sources, but
  - They lack a clear justification in the TBM;
  - They are not associative (to be addressed later).

The distinctness assumption

Definition

- Real-world meaning of this notion difficult to describe
- Main idea: no elementary item of evidence should be counted twice.
  - Example: non overlapping random samples from a population;
  - Counterexample: opinions of different people based on overlapping experiences.
- The TBM conjunctive and disjunctive rules are not appropriate for handling highly overlapping evidence (they are not idempotent).

Relaxing the distinctness assumption

Main approaches

- Possible approaches for combining overlapping items of evidence:
  - Describe the nature of the interaction between sources (Dubois and Prade 1986; Smets 1986);
  - Use a combination rule tolerating redundancy in the combined information.
- Such a rule should be idempotent: $m * m = m$.
- Idempotent rules exist (averaging; Cattaneo, 2003; Destercke et al, 2007), but they are not associative.

The associativity requirement

- Definition: $(m_1 * m_2) * m_3 = m_1 * (m_2 * m_3)$ for all m_1, m_2, m_3.
- Why is associativity a desirable property?
- Practical argument:
  - Items evidence can be combined incrementally and regardless of the order in which they are processed (provided commutativity is also verified);
  - Quasi-associativity (existence of an n-ary operator $op(m_1, \ldots, m_n)$ may be sufficient in that respect.
- Conceptual argument: $m_1 * m_2$ should capture all the relevant information contained in m_1 and m_2; consequently it should not be necessary to keep m_1 and m_2 in memory for further processing.

Main results to be presented in this talk

- Two new idempotent and associative combination rules, applicable to combine possibly overlapping items of evidence:
  - the cautious conjunctive rule $\owedge$
  - the bold disjunctive rule $\ovee$
- These rules are derived from the Least commitment principle (an equivalent of the maximum entropy principle for belief functions).
- Each of the four rules $\bigcirc\!\!\!\!\cap$, $\bigcirc\!\!\!\!\cup$, $\owedge$ and $\ovee$ occupies a special position in a distinct infinite family of rules with identical algebraic properties.

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Basic belief assignment
Definition

Let $\Omega = \{\omega_1, \ldots, \omega_K\}$ be a finite set of answers to a given question Q, called a frame of discernment.

Definition (Basic belief assignment)

A basic belief assignment (BBA) on Ω is a mapping $m : 2^\Omega \rightarrow [0, 1]$ such that

$$\sum_{A \subseteq \Omega} m(A) = 1$$

Subsets A of Ω such that $m(A) > 0$ are called focal sets of m.

Basic belief assignment

Interpretation

- A BBA m represents:
  - the state of knowledge of a rational agent Ag at a given time t, regarding question Q;
  - by extension, an item of evidence that induces such a state of knowledge.
- $m(A)$: part of a unit mass of belief assigned to A and to no strict subset.
- $m(\Omega)$: degree of ignorance.
- $m(\emptyset)$: degree of conflict. Under the open-world assumption, degree of belief in the hypothesis that the true answer to question Q does not lie in Ω.

Associated functions
Belief and implicability functions

Definition (Belief function)

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \quad \forall A \subseteq \Omega$$

Interpretation of $bel(A)$: degree of belief in A.

Definition (Implicability function)

$$b(A) = bel(A) + m(\emptyset), \quad \forall A \subseteq \Omega$$

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Associated functions

Plausibility and commonality

Definition (Plausibility function)

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \quad \forall A \subseteq \Omega$$

Definition (Commonality function)

$$q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega$$

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Equivalence of representations

- Functions *bel*, *b*, *pl*, *q*, *m* are in one-to-one correspondance.
- One can move from any representation to another using linear transformations.
- For instance:

$$pl(A) = bel(\Omega) - bel(\overline{A}) = 1 - b(\overline{A}), \quad \forall A \subseteq \Omega,$$

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Omega,$$

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} b(B), \quad \forall A \subseteq \Omega,$$

- There exists at least two other equivalent representations (to be introduced later...)

TBM conjunctive rule

Definition

Definition (TBM conjunctive rule)

$m_{1 \bigcirc\!\!\!\!\cap\, 2} = m_1 \bigcirc\!\!\!\!\cap\, m_2$ *defined as:*

$$m_{1 \bigcirc\!\!\!\!\cap\, 2}(A) = \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega,$$

Interpretation: $m_1 \bigcirc\!\!\!\!\cap\, m_2$ encodes the agent's belief after receiving m_1 and m_2 from two sources S_1 and S_2, assuming that:

- S_1 and S_2 are distinct (Klawonn and Smets, 1992);
- both S_1 and S_2 are reliable.

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TBM conjunctive rule
Properties

- Algebraic properties:
  - Commutativity,
  - Associativity
  - Neutral element: vacuous BBA m_Ω ($m_\Omega(\Omega) = 1$)

$\rightarrow (\mathcal{M}, \bigcirc\!\!\!\!\cap)$ is a commutative monoid.

- Expression using the commonality functions:

$$q_{1 \bigcirc\!\!\!\!\cap 2}(A) = q_1(A) \cdot q_2(A), \quad \forall A \subseteq \Omega.$$

TBM disjunctive rule
Definition

Definition (TBM disjunctive rule)

$m_{1 \textcircled{\cup} 2} = m_1 \textcircled{\cup} m_2$ *defined as:*

$$m_{1 \textcircled{\cup} 2}(A) = \sum_{B \cup C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega,$$

Interpretation: $m_1 \textcircled{\cup} m_2$ encodes the agent's belief after receiving m_1 and m_2 from two sources S_1 and S_2, assuming that:

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TBM disjunctive rule

Properties

- Algebraic properties:
  - Commutativity,
  - Associativity
  - Neutral element: $m_\emptyset$ ($m_\emptyset(\emptyset) = 1$)

  $\rightarrow$ $(\mathcal{M}, \bigcirc\!\!\!\!\cup)$ is a commutative monoid.
- Expression using the implicability functions:

$$b_{1 \bigcirc\!\!\!\!\cup 2}(A) = b_1(A) \cdot b_2(A), \quad \forall A \subseteq \Omega.$$

Complementation and De Morgan laws

- Complement of m:

$$\overline{m}(A) = m(\overline{A}), \quad \forall A \subseteq \Omega.$$

- De Morgan laws for $\bigcirc\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\cup$:

$$\begin{aligned} \overline{m_1 \bigcirc\!\!\!\!\cup\, m_2} &= \overline{m_1} \bigcirc\!\!\!\!\cap\, \overline{m_2}, \\ \overline{m_1 \bigcirc\!\!\!\!\cap\, m_2} &= \overline{m_1} \bigcirc\!\!\!\!\cup\, \overline{m_2}, \end{aligned}$$

($\bigcirc\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\cup$ can be interpreted as generalized intersection and union)

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Simple BBA
Definition and notation

Definition (Simple BBA)

A BBA is simple if it is of the form

$$m(A) = 1 - w$$
$$m(\Omega) = w,$$

with $w \in [0,1]$ *and* $A \subseteq \Omega$*. Notation:* $m = A^w$.

- Property: $A^{w_1} \cap\!\!\!\!\bigcirc A^{w_2} = A^{w_1 w_2}$.
- Special cases:
  - Vacuous BBA: A^1 with any A.
  - Categorical BBA: A^0.
- Can any BBA be decomposed as the $\cap\!\!\!\!\bigcirc$-combination of simple BBAs?

Simple BBA
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Separable BBA

Definition

- The concept of separability was introduced by Shafer (1976) in the case of normal BBAs. It can be adapted to subnormal BBAs as follows.

Definition (separability)

A BBA m is separable if it can be decomposed as the $\textcircled{\cap}$ *combination of simple BBAs.*

- This decomposition is unique as long as m is nondogmatic ($m(\Omega) > 0$). It may be called the canonical conjunctive decomposition of m.

Separable BBA

Conjunctive weight function

- If m is separable, then there exists a unique function $w : 2^\Omega \mapsto (0,1]$ such that

$$m = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w(A)},$$

  and $w(\Omega) = 1$ by convention.
- Function w is called the conjunctive weight function associated to m. It is thus yet another representation of m.
- Can this representation be extended to any nondogmatic BBA?

Generalized simple BBA
Definition

Definition (Smets, 1995)

A generalized simple BBA is a function $\mu : 2^\Omega \to \mathbb{R}$ such that

$$\begin{aligned} \mu(A) &= 1 - w, \\ \mu(\Omega) &= w, \\ \mu(B) &= 0 \quad \forall B \in 2^\Omega \setminus \{A, \Omega\}, \end{aligned}$$

for some $A \neq \Omega$ and $w \in [0, +\infty)$. Notation: $\mu = A^w$.

Generalized simple BBA
Interpretation

- If $w \leq 1$, μ is a simple BBA.
- If $w > 1$, μ is not a BBA $\rightarrow$ inverse BBA.
- Interpretation : models a state of knowledge in which we have some diffidence (disbelief) against hypothesis A. We need to acquire some evidence in favor of A to reach a neutral state:
$$A^w \textcircled{\cap} A^{1/w} = A^1.$$

Canonical decomposition of a nondogmatic BBA
Main result

Theorem (Smets, 1995)

Any nondogmatic BBA can be uniquely decomposed as the $\bigcirc\!\!\!\!\cap$ *of generalized simple BBAs:*

$$m = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w(A)},$$

with $w(A) \in (0, +\infty[$ *for all* $A \subset \Omega$*.*

- The canonical weight function is now from 2^Ω to $(0, +\infty[$.
- m is separable iff $w(A) \leq 1$ for all A.

Canonical decomposition of a nondogmatic BBA

Main result

Theorem (Smets, 1995)

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- The canonical weight function is now from 2^Ω to $(0, +\infty[$.
- m is separable iff $w(A) \leq 1$ for all A.

Conjunctive weight function
Computation

- Computation of w from q:

$$\ln w(A) = -\sum_{B \supseteq A} (-1)^{|B|-|A|} \ln q(B), \quad \forall A \subset \Omega.$$

- Similarity with

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Omega$$

- Any procedure for transforming q to m can be used to transform $-\ln q$ to $\ln w$.

Examples

Consonant BBAs

- Let m be a consonant BBA, with associated possibility distribution $\pi_k = \pi(\omega_k) = q(\{\omega_k\})$, $k = 1, \ldots, K$, such that

$$1 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_K > 0.$$

- The conjunctive weight function associated to m is:

$$w(A) = \begin{cases} \pi_1 & A = \emptyset, \\ \frac{\pi_{k+1}}{\pi_k}, & A = \{\omega_1, \ldots, \omega_k\},\ 1 \leq k < K, \\ 1, & \text{otherwise.} \end{cases}$$

- m is separable.

Examples

Quasi-Bayesian BBAs

- Let m be a BBA on Ω with focal sets $A_1, \ldots, A_n$, and Ω, such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$.
- We assume that $m(\Omega) + \sum_{k=1}^{n} m(A_k) \leq 1$, so that $\emptyset$ may also be a focal set.
- The conjunctive weight function associated to m is:

$$w(A) = \begin{cases} \frac{m(\Omega)}{m(A_k)+m(\Omega)}, & A = A_k, \\ m(\Omega) \prod_{k=1}^{n} \left(1 + \frac{m(A_k)}{m(\Omega)}\right), & A = \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

- We may have $w(\emptyset) > 1$, so that m is not always separable.

Expression of the TBM conjunctive rule using w

Property

We have

$$
\begin{aligned}
m_1 \textcircled{\cap} m_2 &= \left(\textcircled{\cap}_{A \subset \Omega} A^{w_1(A)} \right) \textcircled{\cap} \left(\textcircled{\cap}_{A \subset \Omega} A^{w_2(A)} \right) \\
&= \textcircled{\cap}_{A \subset \Omega} A^{w_1(A) w_2(A)}.
\end{aligned}
$$

Consequently,

$$
w_{1 \textcircled{\cap} 2} = w_1 \cdot w_2.
$$

- Similar to $q_{1 \textcircled{\cap} 2} = q_1 \cdot q_2$.

Summary

- Several alternative representations of a BBA, including *bel*, *b*, *pl*, *q* and *w*.
- The TBM conjunctive and disjunctive rules are usually expressed in the *m*-space, but they have simpler representations in other spaces:
  - *q* and *w* spaces for $\bigcirc\!\!\!\!\cap$
  - *b* space and another space to be introduced later for $\bigcirc\!\!\!\!\cup$.
- Most attempts to generalize $\bigcirc\!\!\!\!\cap$ have started from its expression in the *m* space.
- Our approach will be based on the *w* space.

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Least commitment principle
Definition

Definition (Least commitment principle)

Given two belief functions compatible with a set of constraints, the most appropriate is the least committed (informative).

- Similar to the maximum entropy principle in Probability theory.
- To make this principle operational, it is necessary to define ways of comparing belief functions according to their information content: "m_1 is more committed than m_2".
- Several such informational orderings have been proposed.

Informational Comparison of Belief Functions
Definitions

pl-ordering: $m_1 \sqsubseteq_{pl} m_2$ iff $pl_1(A) \leq pl_2(A)$, for all $A \subseteq \Omega$;

q-ordering: $m_1 \sqsubseteq_q m_2$ iff $q_1(A) \leq q_2(A)$, for all $A \subseteq \Omega$;

s-ordering: $m_1 \sqsubseteq_s m_2$ iff there exists a stochastic matrix S with general term $S(A, B)$, $A, B \in 2^\Omega$ verifying $S(A, B) > 0 \Rightarrow A \subseteq B$, $A, B \subseteq \Omega$, such that

$$m_1(A) = \sum_{B \subseteq \Omega} S(A, B) m_2(B), \quad \forall A \subseteq \Omega.$$

d-ordering: $m_1 \sqsubseteq_d m_2$, iff there exists a BBA m such that $m_1 = m \bigcirc\!\!\!\!\cap\, m_2$.

Informational Comparison of Belief Functions

Properties

- $m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2, \end{cases}$
- The vacuous BBA m_Ω is the unique greatest element for $\sqsubseteq_x$ with $x \in \{pl, q, s, d\}$:

$$m \sqsubseteq_x m_\Omega, \quad \forall m, \forall x \in \{pl, q, s, d\}.$$

- Monotonicity of $\bigcirc\!\!\!\!\cap$ with respect to $\sqsubseteq_x$, $x \in \{pl, q, s, d\}$:

$$m_1 \sqsubseteq_x m_2 \Rightarrow m_1 \bigcirc\!\!\!\!\cap\, m_3 \sqsubseteq_x m_2 \bigcirc\!\!\!\!\cap\, m_3, \quad \forall m_1, m_2, m_3$$

$\rightarrow$ $(\mathcal{M}, \bigcirc\!\!\!\!\cap, \sqsubseteq_x)$ is a partially ordered commutative monoid.

Cautious combination of belief functions

Principle (Dubois, Prade and Smets, 2001)

- Two sources provide BBAs m_1 and m_2, and the sources are both considered to be reliable.
- The agent's state of belief, after receiving these two pieces of information, should be represented by a BBA m_{12} more committed than m_1, and more committed than m_2.
- Let $\mathcal{S}_x(m)$ be the set of BBAs m' such that $m' \sqsubseteq_x m$, for some $x \in \{pl, q, s, d\}$.
- We thus have $m_{12} \in \mathcal{S}_x(m_1)$ and $m_{12} \in \mathcal{S}_x(m_2)$ or, equivalently, $m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$.
- According to the LCP, one should select the x-least committed element in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$, if it exists.

Cautious combination of belief functions

Problem

- The above approach works for special cases.
- Example (Dubois, Prade, Smets 2001): if m_1 and m_2 are consonant, then the q-least committed element in $\mathcal{S}_q(m_1) \cap \mathcal{S}_q(m_2)$ exists and it is unique: it is the consonant BBA with commonality function $q_{12} = q_1 \wedge q_2$.
- In general, neither existence nor unicity of a solution can be guaranteed with any of the x-orderings, $x \in \{pl, q, s, d\}$.
- We need to define a new ordering relation.

The w-ordering
Definition and properties

Definition (w-ordering)

Let m_1 and m_2 be two nondogmatic BBAs.
$m_1 \sqsubseteq_w m_2$ iff $w_1(A) \leq w_2(A)$, for all $A \subset \Omega$.

- Interpretation: $m_1 = m \bigcirc\!\!\!\!\cap\, m_2$ for some separable BBA m.
- $m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2, \end{cases}$
- No greatest element, but m_Ω is the unique maximal element: $m_\Omega \sqsubseteq_w m \Rightarrow m = m_\Omega$.
- Monotonicity of $\bigcirc\!\!\!\!\cap$:
  $m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \bigcirc\!\!\!\!\cap\, m_3 \sqsubseteq_w m_2 \bigcirc\!\!\!\!\cap\, m_3, \quad \forall m_1, m_2, m_3$

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The cautious conjunctive rule
Definition

Theorem

Let m_1 and m_2 be two nondogmatic BBAs. The w-least committed element in $\mathcal{S}_w(m_1) \cap \mathcal{S}_w(m_2)$ exists and is unique. It is defined by the following weight function:

$$w_{1 \owedge 2}(A) = w_1(A) \wedge w_2(A), \quad \forall A \subset \Omega.$$

Definition (cautious conjunctive rule)

$$m_1 \owedge m_2 = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \wedge w_2(A)}.$$

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The cautious conjunctive rule
Computation

Cautious rule computation

m-space		w-space
m_1	$\longrightarrow$	w_1
m_2	$\longrightarrow$	w_2
$m_1 \textcircled{\wedge} m_2$	$\longleftarrow$	$w_1 \wedge w_2$

The cautious conjunctive rule
Properties

Commutativity: $\forall m_1, m_2,\ m_1 \owedge m_2 = m_2 \owedge m_1$

Associativity: $\forall m_1, m_2, m_3$,

$$m_1 \owedge (m_2 \owedge m_3) = (m_1 \owedge m_2) \owedge m_3$$

No neutral element: $m_\Omega \owedge m = m$ iff m is separable.

Monotonicity:

$$m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \owedge m_3 \sqsubseteq_w m_2 \owedge m_3, \quad \forall m_1, m_2, m_3.$$

$\rightarrow$ $(\mathcal{M}_{nd}, \owedge, \sqsubseteq_w)$ is a partially ordered commutative semigroup.

The cautious conjunctive rule

Properties related to the combination of non distinct evidence

Idempotence: $\forall m, \ m \circledwedge m = m$

Distributivity $\textcircled{\cap}$ with respect to $\circledwedge$:

$$(m_1 \textcircled{\cap} m_2) \circledwedge (m_1 \textcircled{\cap} m_3) = m_1 \textcircled{\cap} (m_2 \circledwedge m_3), \quad \forall m_1, m_2, m_3.$$

$\rightarrow$ Item of evidence m_1 is not counted twice!

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Bold disjunctive combination of belief functions
Principle

- The agent receives two BBAs m_1 and m_2 from two sources, at least one of which is considered to be reliable.
- The resulting BBA should be less committed than m_1 and m_2.
- Formally, $m_{12} \in \mathcal{G}_x(m_1) \cap \mathcal{G}_x(m_2)$, for some $x \in \{w, d, s, pl, q\}$, with $\mathcal{G}_x(m)$= set of BBAs less committed than m according to $\sqsubseteq_x$.
- Most commitment principle: we should choose in $\mathcal{G}_x(m_1) \cap \mathcal{G}_x(m_2)$ the most committed BBA according to $\sqsubseteq_x$ (if it exists).

Bold disjunctive combination of belief functions
Search for a suitable informational oredring

- With $x = w$, this approach leads to a mass function m_{12} defined by $w_{12} = w_1 \vee w_2$.
- OK with separable BBAs, but $w_1 \vee w_2$ does not always correspond to a belief function.
- We need yet another ordering relation...

Canonical disjunctive decomposition

Principle

- Let m be a subnormal BBA. Its complement $\overline{m}$ is nondogmatic and can be decomposed as

$$\overline{m} = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{\overline{w}(A)}.$$

- Consequently, m can be written

$$m = \overline{\bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{\overline{w}(A)}} = \bigcirc\!\!\!\!\cup_{A \subset \Omega} \overline{A^{\overline{w}(A)}}.$$

- Each BBA $\overline{A^{\overline{w}(A)}}$ is the complement of a generalized simple BBA. Its focal sets are $\overline{A}$ and $\emptyset$. Notation: $\overline{A}_{v(\overline{A})}$, with $v(\overline{A}) = \overline{w}(A)$.

Canonical disjunctive decomposition

Disjunctive weight function

Theorem

Any subnormal BBA m can be uniquely decomposed as the $\textcircled{\cup}$-combination of generalized BBAs $A_{v(A)}$ assigning a mass $v(A) > 0$ to $\emptyset$, and a mass $1 - v(A)$ to A, for all $A \subseteq \Omega$, $A \neq \emptyset$:

$$m = \textcircled{\cup}_{A \neq \emptyset} A_{v(A)}. \quad (1)$$

Definition (Disjunctive weight function)

Function $v : 2^\Omega \setminus \{\emptyset\} \rightarrow (0, +\infty)$ will be referred to as the disjunctive weight function.

Disjunctive weight function

Properties

- Duality with w: $v(A) = \overline{w}(\overline{A}), \forall A \neq \emptyset$ (similar to $b(A) = \overline{q}(\overline{A})$).
- Computation from b:

$$\ln v(A) = -\sum_{B \subseteq A} (-1)^{|A|-|B|} \ln b(B).$$

  Similarity with

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} b(B), \quad \forall A \subseteq \Omega.$$

- TBM disjunctive rule:

$$v_{1 \bigcirc\!\!\!\!\cup 2} = v_1 \cdot v_2.$$

The v-ordering
Definition and properties

Definition (v-ordering)

Let m_1 and m_2 be two subnormal BBAs.
$m_1 \sqsubseteq_v m_2$ iff $v_1(A) \geq v_2(A)$, for all $A \neq \emptyset$.

- Interpretation: $m_2 = m \bigcirc\!\!\!\!\cup\, m_1$ for some BBA m such that $\overline{m}$ is separable.
- $m_1 \sqsubseteq_v m_2 \Rightarrow m_1 \sqsubseteq_s m_2$.
- No smallest element, but $m_\emptyset$ is the unique minimal element: $m \sqsubseteq_v m_\emptyset \Rightarrow m = m_\emptyset$.
- Monotonicity of $\bigcirc\!\!\!\!\cup$:
  $m_1 \sqsubseteq_v m_2 \Rightarrow m_1 \bigcirc\!\!\!\!\cup\, m_3 \sqsubseteq_v m_2 \bigcirc\!\!\!\!\cup\, m_3, \quad \forall m_1, m_2, m_3$

The v-ordering
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The bold disjunctive rule

Definition

Theorem

Let m_1 and m_2 be two subnormal BBAs. The v-most committed element in $\mathcal{G}_v(m_1) \cap \mathcal{G}_v(m_2)$ exists and is unique. It is defined by the following disjunctive weight function:

$$v_{1 \textcircled{\vee} 2}(A) = v_1(A) \wedge v_2(A), \quad \forall A \in 2^\Omega \setminus \emptyset.$$

Definition (Bold disjunctive rule)

$$m_1 \textcircled{\vee} m_2 = \textcircled{\cup}_{A \neq \emptyset} A_{v_1(A) \wedge v_2(A)}.$$

The bold disjunctive rule
Computation

Bold rule computation

m-space		v-space
m_1	$\longrightarrow$	v_1
m_2	$\longrightarrow$	v_2
$m_1 \textcircled{\vee} m_2$	$\longleftarrow$	$v_1 \wedge v_2$

The bold disjunctive rule
Properties

Commutativity: $\forall m_1, m_2,\ m_1 \circledvee m_2 = m_2 \circledvee m_1$

Associativity: $\forall m_1, m_2, m_3,\ m_1 \circledvee (m_2 \circledvee m_3) = (m_1 \circledvee m_2) \circledvee m_3$

No neutral element: $m_\emptyset \circledvee m = m$ iff $\overline{m}$ is separable.

Monotonicity:

$$m_1 \sqsubseteq_v m_2 \Rightarrow m_1 \circledvee m_3 \sqsubseteq_v m_2 \circledvee m_3, \quad \forall m_1, m_2, m_3.$$

$\rightarrow$ $(\mathcal{M}_s, \circledvee, \sqsubseteq_v)$ is a partially ordered commutative semigroup.

The bold disjunctive rule
Properties (continued)

Idempotence: $\forall m,\ m \textcircled{\vee} m = m$;

Distributivity of $\textcircled{\cup}$ with respect to $\textcircled{\vee}$:

$$(m_1 \textcircled{\cup} m_2) \textcircled{\vee} (m_1 \textcircled{\cup} m_3) = m_1 \textcircled{\cup} (m_2 \textcircled{\vee} m_3), \quad \forall m_1, m_2, m_3.$$

$\rightarrow$ Item of evidence m_1 is not counted twice.

De Morgan laws:

$$\overline{m_1 \textcircled{\vee} m_2} = \overline{m}_1 \textcircled{\wedge} \overline{m}_2$$
$$\overline{m_1 \textcircled{\wedge} m_2} = \overline{m}_1 \textcircled{\vee} \overline{m}_2$$

Generalizing the cautious and bold rules

Four basic rules

	product	minimum	$*$
conjunctive weights w	$\bigcirc\!\!\!\!\cap$	$\bigcirc\!\!\!\!\wedge$	?
disjunctive weights v	$\bigcirc\!\!\!\!\cup$	$\bigcirc\!\!\!\!\vee$	?

- Properties of the minimum and the product on $(0, +\infty]$:
  - Commutativity, associativity;
  - Monotonicity: $x \leq y \Rightarrow x \wedge z \leq y \wedge z, \forall x, y, z \in (0, +\infty]$.
- Neutral element:
  - $+\infty$ for the minimum $\rightarrow$ t-norm;
  - 1 for the product $\rightarrow$ uninorm.
- Generalization to other t-norms and uninorms?

Overview

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2. The cautious and bold rules
   - Informational orderings and the LCP
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   - The bold disjunctive rule
3. Families of combination rules
   - T-norm-based rules
   - Uninorm-based rules
   - Applications

T-norm based conjunctive rules

Definition

Proposition

Let $*$ be a positive t-norm on $(0, +\infty]$. Then, for any conjunctive weight functions w_1 and w_2, the function w_{1*2} defined by :

$$w_{1*2}(A) = w_1(A) * w_2(A), \forall A \subset \Omega,$$

is a conjunctive weight function associated to some nondogmatic BBA m_{1*2}.

Definition (T-norm-based conjunctive rule)

$$m_1 \circledast_w m_2 = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) * w_2(A)}.$$

T-norm based conjunctive rules
Properties

- Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, and $\circledast_w$ the conjunctive rule based on t-norm $*$. Then $(\mathcal{M}_{nd}, \circledast_w, \sqsubseteq_w)$ is a commutative, partially ordered semigroup.
- The minimum is the largest t-norm on $(0, +\infty]$. Consequently:

Proposition

Among all t-norm based conjunctive operators, the cautious rule is the w-least committed:

$$m_1 \circledast_w m_2 \sqsubseteq_w m_1 \owedge m_2, \quad \forall m_1, m_2.$$

T-norm based disjunctive rules

Definition and properties

- Let $*$ be a t-norm on $(0, +\infty]$. The disjunctive rule asociated to $*$ is

$$m_1 \circledast_v m_2 = \bigcup_{\emptyset \neq A \subseteq \Omega} A_{v_1(A) * v_2(A)}.$$

- $(\mathcal{M}_s, \circledast_v, \sqsubseteq_v)$ is a commutative, partially ordered semigroup.
- Among all t-norm based disjunctive operators, the bold rule is the v-most committed.
- De Morgan laws:

$$\begin{aligned} \overline{m_1 \circledast_w m_2} &= \overline{m}_1 \circledast_v \overline{m}_2 \\ \overline{m_1 \circledast_v m_2} &= \overline{m}_1 \circledast_w \overline{m}_2 \end{aligned}$$

Construction of t-norms on $(0, +\infty]$

Proposition

Let $\top$ be a positive t-norm on $[0,1]$, and let $\perp$ be a t-conorm on $[0,1]$. Then the operator $*_{\top,\perp}$ defined by

$$x *_{\top,\perp} y = \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(\frac{1}{x} \perp \frac{1}{y}\right)^{-1} & \text{if } x \wedge y > 1, \\ x \wedge y & \text{otherwise,} \end{cases}$$

for all $x, y \in (0, +\infty]$ is a t-norm on $(0, +\infty]$.

$\rightarrow$ For each pair $(\top, \perp)$, there is a pair of dual conjunctive and disjunctive rules generalizing the cautious and bold rules, respectively.

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Uninorm-based conjunctive rules

Definition

Proposition

Let $\circ$ be a uninorm on $(0,+\infty]$ with 1 as neutral element, such that $x \circ y \leq xy$ for all $x, y \in (0,+\infty]$. Then, for any w functions w_1 and w_2, the function $w_{1\circ 2}$ defined by :

$$w_{1\circ 2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega,$$

is a w function associated to some nondogmatic BBA $m_{1\circ 2}$.

Definition (Uninorm-based conjunctive rule)

Let $\circ$ be a uninorm on $(0,+\infty]$ verifying the above condition.

$$m_1 \circledcirc_w m_2 = \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{w_1(A)\circ w_2(A)}.$$

Uninorm-based conjunctive rules
Properties

Proposition

Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, and $\circledcirc_w$ the conjunctive rule based on uninorm $\circ$ with one as neutral element, and verifying $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. Then $(\mathcal{M}_{nd}, \circledcirc_w, \sqsubseteq_w)$ is a commutative, partially ordered monoid, with the vacuous BBA as neutral element.

- Question: Can we relax the condition $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$, and get an operator $\circledcirc_w$ that is not more committed than $\textcircled{\cap}$?

Uninorm-based conjunctive rules
Properties (continued)

Theorem (Pichon and Denœux, 2007)

Let $\circ$ *be a binary operator on* $(0, +\infty]$ *such that*

- $x \circ 1 = 1 \circ x = x$ *for all* x *and*
- $x \circ y > xy$ *for some* $x, y > 0$.

Then, there exists two BBAs m_1 *and* m_2 *such that* $w_1 \circ w_2$ *is not a valid* w *function.*

Corollary

Consequence: among all uninorm-norm based conjunctive operators, the TBM conjunctive rule is the w*-least committed:*

$$m_1 \circledcirc_w m_2 \sqsubseteq_w m_1 \textcircled{\cap} m_2, \quad \forall m_1, m_2, \forall \circledcirc_w .$$

Uninorm-based disjunctive rules

Definition and properties

- Let $\circ$ be a uninorm on $(0, +\infty]$ with 1 as neutral element, such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. The disjunctive rule associated to $\circ$ is defined as:

$$m_1 \circledcirc_v m_2 = \bigcup\!\!\!\!\bigcirc_{A \subset \Omega} A_{v_1(A) \circ v_2(A)}.$$

- $(\mathcal{M}_s, \circledcirc_v, \sqsubseteq_v)$ is a commutative, partially ordered monoid, with $m_\emptyset$ as neutral element.
- Among all uninorm-norm based disjunctive operators, the TBM disjunctive rule is the v-most committed.
- De Morgan laws:

$$\overline{m_1 \circledcirc_w m_2} = \overline{m}_1 \circledcirc_v \overline{m}_2$$

$$\overline{m_1 \circledcirc_v m_2} = \overline{m}_1 \circledcirc_w \overline{m}_2$$

Construction of uninorms on $(0, +\infty]$

Proposition

Let $\top$ be a positive t-norm on $[0,1]$ verifying $x \top y \leq xy$ for all $x, y \in [0,1]$, and let $\top'$ be a t-norm on $[0,1]$ verifying $x \top y \geq xy$ for all $x, y \in [0,1]$. Then the operator defined by

$$x \circ_{\top,\top'} y = \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(\frac{1}{x} \top' \frac{1}{y}\right)^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases}$$

for all $x, y \in (0, +\infty]$ is a uninorm on $(0, +\infty]$ verifying $x \circ_{\top,\top'} y \leq xy$ for all $x, y > 0$.

→ For each pair $(\top, \top')$, there is a pair of dual conjunctive and disjunctive uninorm-based rules.

Coincidence for separable BBAs

- Let $\top$ and $\top'$ be t-norms on $[0, 1]$, and $\perp$ be a t-conorm on $[0, 1]$.
- One can build:
  - a t-norm $*_{\top,\perp}$ on $(0, +\infty]$;
  - a uninorm $\circ_{\top,\top'}$ on $(0, +\infty]$.
- The corresponding t-norm and uninorm based conjunctive rules $\circledast_w$ and $\circledcirc_w$ coincide on separable BBAs.
- Consequence: to define a rule for combining separable BBAs, one only needs to define a t-norm $\top$.

Summary

- We now have four infinite families of rules:
  - conjunctive and disjunctive t-norm-based rules;
  - conjunctive and disjunctive uninorm-based rules.
- In each of these families, one rule plays a special role and is well justified by the LCP:
  - the $\textcircled{\wedge}$ and $\textcircled{\cap}$ rules are the w-least-committed conjunctive rules in the t-norm-based and uninorm-based families, respectively;
  - the $\textcircled{\vee}$ and $\textcircled{\cup}$ rules are the v-most committed disjunctive rules in the t-norm-based and uninorm-based families, respectively.
- The justification of the other rules is less clear but...
- Can they be useful in practice?

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Application to classification
The problem

- Let us consider a classification problem where objects are described by feature vectors $\mathbf{x} \in \mathbb{R}^p$ and belong to one of K groups in $\Omega = \{\omega_1, \ldots, \omega_K\}$.
- Learning set $\mathcal{L} = \{(\mathbf{x}_1, z_1), \ldots, (\mathbf{x}_n, z_n)\}$, where $z_i \in \Omega$ denotes the class of object i.
- Problem: predict the class of a new object described by feature vector $\mathbf{x}$.
- Application of new combination rules to:
  - combine neighborhood information in the evidential k nearest neighbor rule;
  - combine outputs from classifiers built from different features.

Example 1: evidential k-NN rule

Principle

- The evidence of example i is represented by a simple BBA m_i on Ω defined by

$$m_i = \{z_i\}^{\varphi(d_i)}$$

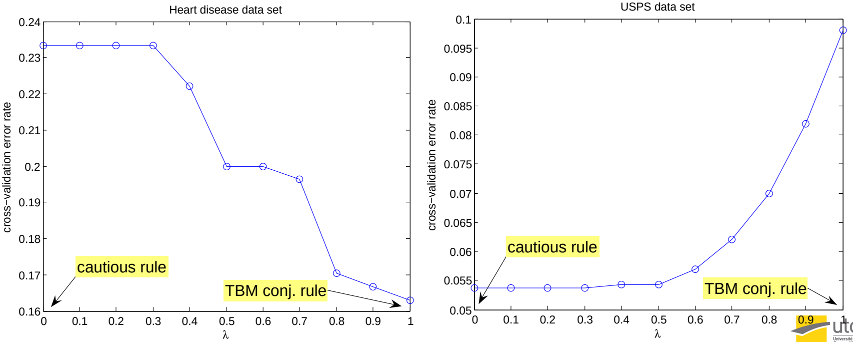
where d_i is the distance between $\mathbf{x}$ and $\mathbf{x}_i$, and φ is an increasing function from $\mathbb{R}^+$ to $[0, 1]$.

- The evidence of the k nearest neighbors of $\mathbf{x}$ in $\mathcal{L}$ is pooled using the TBM conjunctive rule:

$$m = \bigcirc\!\!\!\!\cap_{i \in N_k(\mathbf{x})} \{z_i\}^{\varphi(d_i)}.$$

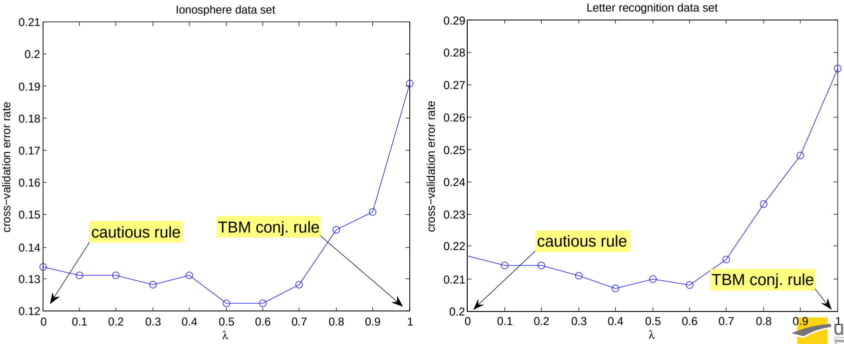
- Generalization: replace $\bigcirc\!\!\!\!\cap$ by another conjunctive operator $\circledast_w$ defined by a t-norm taken in a parameterized family ranging from the product to the minimum (e.g. Dubois-Prade, Frank).

Results
Heart disease and USPS datasets



Results

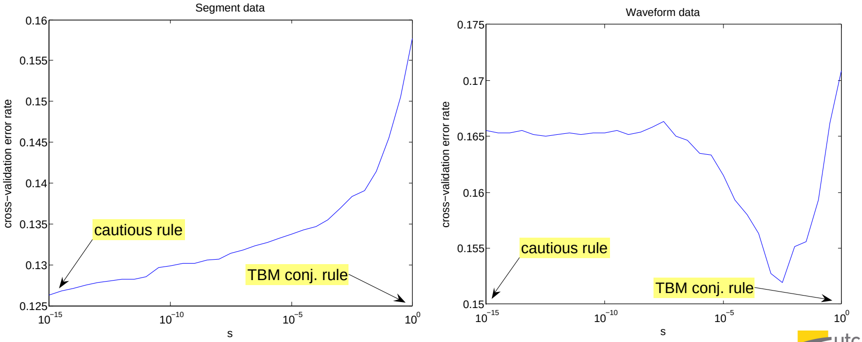
Ionosphere and Letter recognition datasets



Example 2: classifier fusion
Principle

- One separate classifier for each feature $\mathbf{x}_j$.
- Classifier using input feature $\mathbf{x}_j$ produces a BBA m_j.
- Method:
  - logistic regression;
  - posterior probabilities tranformed into consonant BBAs using the isopignistic transformation.
- Classifier outputs combined using t-norm based conjunctive operators.
- T-norm on $[0, 1]$ taken in Frank's family.

Results


Segment data
cross−validation error rate
0.16
0.155
0.15
0.145
0.14
0.135
0.13
0.125
cautious rule
TBM conj. rule
10^{-15}
10^{-10}
10^{-5}
10^{0}
s
Waveform data
cross−validation error rate
0.175
0.17
0.165
0.16
0.155
0.15
cautious rule
TBM conj. rule
10^{-15}
10^{-10}
10^{-5}
10^{0}
s

Summary
Four basic rules

- Two new dual commutative, associative et idempotent rules:
  - cautious conjunctive rule $w_{1 \textcircled{\wedge} 2} = w_1 \wedge w_2$;
  - bold disjunctive rule $v_{1 \textcircled{\vee} 2} = v_1 \wedge v_2$.
- Both rules are derived from the Least commitment principle, for some (different) informational ordering relations.
- With the TBM conjunctive and disjunctive rules, we now have four basic rules:

sources	all reliable	at least one reliable
distinct	$\textcircled{\cap}$	$\textcircled{\cup}$
non distinct	$\textcircled{\wedge}$	$\textcircled{\vee}$

Summary
Algebraic properties

- The $\textcircled{\wedge}$ and $\textcircled{\cap}$ rules have fundamentally different algebraic properties:
  - the $\textcircled{\wedge}$ rule is based on a t-norm on $(0, +\infty]$ and has no neutral element;
  - the $\textcircled{\cap}$ rule is based on a uninorm on $(0, +\infty]$ and has a neutral element (the vacuous BBA).
- Similarly, the $\textcircled{\vee}$ and $\textcircled{\cup}$ rules are based, respectively, on a t-norm and a uninorm; $\textcircled{\cup}$ has a neutral element, whereas $\textcircled{\vee}$ has not.
- The pairs $\textcircled{\wedge}$-$\textcircled{\vee}$ and $\textcircled{\cap}$-$\textcircled{\cup}$ are dual to each other and are related by De Morgan laws.

Summary

T-norm and uninorm-based rules

- To each of the four basic rules corresponds one infinite family of combination rules:
  - the t-norm-based conjunctive and disjunctive families;
  - the uninorm-based conjunctive and disjunctive families.

  $\rightarrow$ at least as much flexibility and diversity as in Possibility theory!
- Each of the four basic rules occupies a special position in its family:
  - The $\textcircled{\wedge}$ and $\textcircled{\cap}$ rules are the least committed elements;
  - The $\textcircled{\vee}$ and $\textcircled{\cup}$ rules are the most committed elements.
- Preliminary experiments suggest that the use of general t-norm and uninorm-based rules may improve the performances of information fusion systems.

References

Ph. Smets.
The canonical decomposition of a weighted belief.
In *Int. Joint Conf. on Artificial Intelligence*, pages 1896–1901, San Mateo, Ca, 1995. Morgan Kaufman.

T. Denœux.
Conjunctive and Disjunctive Combination of Belief Functions Induced by Non Distinct Bodies of Evidence.
Artificial Intelligence (In press), 2007.